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AUTOMORPHISMS OF THE MODULE FORMAL SUMS

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Dedicated to Professor Mirjana Vuković on the occasion of her 70th birthday

ABSTRACT. Besides Tychonoff topology on a direct product of discreet skew fields a variety of other topologies are defined and studied which correspond to given filters on the support. The main result of this paper is the description of continuous linear operators of a space of the formal sums.

1. MODULE OF FORMAL SUMS

Let K be a field with discreet topology, G be a multiplicative group. Consider linear space $\mathcal{F}(G,K)$ of all maps $G \to K$. The element of this space we will write as formal sum

$$\gamma = \sum_{g\in G} g k_g, \quad (k_g \in K). \tag{1.1}$$

implying here $\gamma(g) = k_g$. The operations of addition and multiplication by elements of field K are pointwise, namely, $(\gamma+\beta)(g) = \gamma(g)+\beta(g)$ $(\gamma\cdot k)(g) = \gamma(g)k$ for any $\gamma,\beta \in \mathcal{F}(G,K)$ $g \in G$, $k \in K$. A set $\{g \in G \mid \gamma(g) \neq 0\}$ is called *a support of a the sum* γ and is denoted by $\operatorname{supp}\gamma$.

Transform the space $\mathcal{F}(G,K)$ into the left G-module, assuming $(h\gamma)(g) = \gamma(h^{-1}g)$ or, more simply $h\cdot\sum_{g\in G} gk_g = \sum_{g\in G}(hg)k_g$. Using distributivity we extend this multiplication by the group ring KG and obtain the left KG-module of the formal sums. Note that the group ring KG is considered as a subspace of the space $\mathcal{F}(G,K)$, consisting of all formal sums with finite support.

The subset $P \subseteq G$ closed with respect to multiplication such that $P\cup P^{-1} = G$ and $P\cap P^{-1} = \{1\}$ is called a *cone in group*. The relation $x \le y \Leftrightarrow x^{-1}y \in P$ transform G into a linearly ordered set and, moreover, into a left-ordered group that is $g_1 \le g_2$ then $gg_1 \le gg_2$ for any $g_1, g_2, g \in G$.
Further we assume that a group G has a cone P and above mentioned linear order.

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The meaning of the notation $G, P, K, \leq$ is preserved throughout the paper. The identity 1 denotes a neutral element with respect to multiplication in any algebraic system.

Suppose that a support of the sum (1.1) has the smallest element g_0. The element g_0 is called *a norm* of the sum of γ and is denoted by $v(\gamma)$. The homogeneous summand $g_0 k_{g_0}$ in γ we denote by $\partial\gamma$. It is clear that this homogeneous summand characterized by the property $\operatorname{supp}(\gamma - \partial\gamma) > v(\gamma)$. (The inequality "$S_1 > S_2$" for subsets S_1, S_2 of any linearly ordered set we will use in the sense that any element from the left part is strictly greater than any element from the right part.) We attach ∞ to the linearly ordered set G, assuming that $g < \infty$ for any $g \in G$ and $v(0) = \infty$.

A formal sum (1.1) is called *a formal series*, if the set $\operatorname{supp}\gamma$ is well-ordered in ascending order, that is any subset of this set has the smallest element. A set of all formal series we denote by $K\{G\}$. Since G is left-ordered group the set of formal series forms a left KG-submodule of the left KG-module $\mathcal{F}(G,K)$.

We topologize the module of formal series by defining the basis of neighborhoods of zero as follows. Consider a well-ordered in descending order subset $\Delta \subseteq G$ that is any non-empty subset has the largest element and define subspace $U(\Delta) = \{\gamma \in K\{G\} \mid \operatorname{supp}\gamma \cap \Delta = \emptyset\}$ of K-space $K\{G\}$. The collection of all such subspaces form the basis of the neighborhoods of zero and give a linear Hausdorff topology $\mathcal{T}$, with respect to which the subspace KG is densely located in $K\{G\}$.

As in any topological group the summation of the collection $\{\gamma_i \mid i \in I\}$ of the formal series is defined. The sum $\sum_{i\in I}\gamma_i$ is equal to the formal series β if for any neighborhood of zero U there is a finite set of indexes $D(U) \subseteq I$ such that for any finite subset $D(U) \subseteq D \subseteq I$ there is a relationship $\sum_{i\in D}\gamma_i \in \beta + U$. The existence of such sum is equivalent to the following two conditions:

1) for any element $g \in G$ the set of indexes $i \in I$ such that $\gamma_i(g) \neq 0$ is finite;

2) the set $\bigcup_{i\in I}\operatorname{supp}\gamma_i$ is well-ordered in ascending order.

Under these conditions, the sum β exists and $\beta(g) = \sum_{i\in I}\gamma_i(g)$ for any $g \in G$. We denote this sum by $\beta = \sum^{\mathcal{T}}_{i\in I}\gamma_i$. In particular, if for a formal series(1.1) we take the group G as a set of indices and put $\gamma_g = gk_g$ then the family (gk_g) summable and its sum is γ. In other words,$\gamma = \sum^{\mathcal{T}}_{g\in G} gk_g$, i.e. the set G is the topological basis of the space $K\{G\}$.

A summability of the family of the formal series is easily checked using the following proposition.

Proposition 1.1. *The system of the formal power series $\gamma_i \in K\{G\}$ ($i \in I$) is $\mathcal{T}$-summable if and only if for any $\mathbb{N}$-sequence $i_1, i_2, i_3, \dots$ of pairwise distinct indices and any set of the elements $g_j \in \operatorname{supp}\gamma_{i_j}$ ($j = 1,2,3,\dots$) there is a pair of natural numbers $j < k$ such that $g_j < g_k$.*

Proof. Since every support $\operatorname{supp}\gamma_i$ is well-ordered set we need to prove that the union $M = \bigcup_{i\in I}\operatorname{supp}\gamma_i$ is well-ordered set if and only if for any $\mathbb{N}$-sequence

$i_1, i_2, i_3, \ldots$ of pairwise distinct indices and any set of the elements $g_j \in \operatorname{supp}\gamma_{i_j}$ $(j = 1, 2, 3, \ldots)$ there is a pair of natural numbers $j, k \in \mathbb{N}$ $j < k$ such that $g_j < g_k$. The implication $\wr \Rightarrow \wr$ is obvious. Let prove the implication $\wr \Leftarrow \wr$. Assume that there exist a $\mathbb{N}$-sequence of pairwise distinct indices $i_1, i_2, i_3, \ldots$ such that for some set of elements $g_j \in \operatorname{supp}\gamma_{i_j}$ there is an infinite chain of inequalities $g_1 \geq g_2 \geq g_3 \geq \ldots$. Since the system $\{\gamma_i\}$ is summable we can assume that all inequalities are strict. But then the union of M is not a well-ordered set. □

2. $\mathcal{T}$-BASES

Let $\{\gamma_i \,|\, i \in I\}$ be a collection of the formal power series. We say that the collection is σ-*summable* if for any well-ordered set $M \subseteq G$ subset $\{\gamma_i \,|\, v(\gamma_i) \in M\}$ is $\mathcal{T}$-summable. A collection $\{\gamma_i \,|\, i \in I\}$ is called *locally complete* if for any $h \in G$ there exist γ_i with norm h and *monotone* if the inequality indexes $i \neq k$ implies inequality of norms $v(\gamma_i) \neq v(\gamma_k)$. In the case of monotone collection we can assume that the set of indices I is a subset of the linearly ordered set G and by reindexing can be achieved that a larger index corresponds to a larger norm.

We say that the collection $\{\gamma_i \,|\, i \in I\}$ is σ-*linearly independent* if for any well-ordered subset $M \subseteq G$ and any set of coefficients $\{k_i \in K \,|\, i \in I\}$ there is an implication

$$\sum_{v(\gamma_i) \in M}^{\mathcal{T}} \gamma_i k_i = 0 \;\Rightarrow\; (\forall i \in I)\, k_i = 0.$$

Note that monotonicity implies σ-linear independence. Analogues of the following theorem can be found [5] and [6].

Theorem 2.1. *Let $\{\gamma_i \,|\, i \in I\}$ be a σ-summable and locally complete collection of the formal power series. Then for any formal series $\gamma \in K\{G\}$ there exist $k_i \in K$ $(i \in I)$ such that*

$$\gamma = \sum_{i \in I}^{\mathcal{T}} \gamma_i k_i. \tag{2.1}$$

If moreover $\{\gamma_i \,|\, i \in I\}$ is σ-linearly independent then the coefficients k_i in the decomposition (2.1) are unique.

Proof. The second part of the theorem is trivially. We prove the first part assuming that $\gamma \neq 0$.

Using the transfinite induction we will choose index $i(\lambda) \in I$ and the coefficient k_λ such that for any ordinals λ_1, λ_2, with $\lambda_1 < \lambda_2 \leq \lambda$ there are relations:

$$v(\gamma_{i(\lambda_1)}) < v(\gamma_{i(\lambda_2)}), \tag{2.2}$$

$$\{\gamma_{i(\mu)} \,|\, \mu < \lambda_2\} \text{ is } \mathcal{T}-\text{summable and } v(\gamma - \sum_{\mu < \lambda_2}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu) = v(\gamma_{i(\lambda_2)}). \tag{2.3}$$

Base induction ($\lambda = 1$). Using completeness, we find for the norm $v(\gamma)$ the index $i(1)$ such that $v(\gamma_{i(1)}) = v(\gamma)$. Then $\partial(\gamma_{i(1)} k_1) = \partial\gamma$ for suitable $k_1 \in K$.

Induction transition. Assume that for all ordinals μ less than λ indices $i(\mu)$ and the coefficients k_μ, satisfying (2.2) and (2.3) for $\lambda_1 < \lambda_2 \leq \mu$ are chosen.

First we will show that the family $\{\gamma_{i(\mu)} \,|\, \mu < \lambda\}$ is $\mathcal{T}$-summable. If λ is nonlimit ordinal then adding a finite set of formal power series to $\mathcal{T}$-summable family we get again the $\mathcal{T}$-summable family. If λ is a limit ordinal, then (2.2) is true for any $\lambda_1 < \lambda_2$. Therefore since the family $\{\gamma_i\}$ is σ-summable, it follows $\mathcal{T}$-summability of the family $\{\gamma_{i(\mu)} \,|\, \mu < \lambda\}$.

If $\gamma = \sum_{\mu<\lambda}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu$ then the first statement of the theorem is proved and the induction construction is completed. Suppose that

$$\beta = \gamma - \sum_{\mu<\lambda}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu \neq 0.$$

Then, as above, we find the index $i(\lambda) \in I$ and the element $k_\lambda \in K$ such that $\partial(\gamma_{i(\lambda)} k_\lambda) = \partial\beta$.

Hence the equality (2.3) takes place for all $\lambda_2 \leq \lambda$. It remains to check (2.2) only for $\lambda_2 = \lambda$.

Case 1. λ *is the limit ordinal.* Denote by $\tau = \lambda_1 + 1$ and note that this ordinal is less than λ. We have:

$$\beta = \gamma - \sum_{\mu<\lambda}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu = \left(\gamma - \sum_{\mu<\tau}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu\right) - \sum_{\tau\leq\mu<\lambda}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu. \tag{2.4}$$

The difference in parentheses has a norm equal to $v(\gamma_{i(\tau)})$ (see (2.3)) which is strictly greater than $v(\gamma_{i(\lambda_1)})$. Each of the summands in the second sum in (2.4) has a norm greater than $v(\gamma_{i(\lambda_1)})$ too (again use (2.2)). Therefore $v(\gamma_{i(\lambda)}) = v(\beta) > v(\gamma_{i(\lambda_1)})$.

Case 2. $\lambda = \tau + 1$ *is the nonlimit ordinal.* Then

$$v(\gamma_{i(\lambda)}) = v(\beta) = v\left(\gamma - \sum_{\mu<\tau}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu - \gamma_{i(\tau)} k_\tau\right) > v(\gamma_{i(\tau)} k_\tau) = v(\gamma_{i(\tau)}),$$

because

$$\partial(\gamma - \sum_{\mu<\tau}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu) = \partial\gamma_{i(\tau)} k_\tau$$

therefore, the difference of these two formal power series has a norm greater than $v(\gamma_{i(\tau)})$. So we checked the inequality (2.2).

Continuing the induction process we get the ordinal λ such that $\gamma = \sum_{\mu<\lambda}{}^{\mathcal{T}} \gamma_{i(\mu)} k_\mu$. This follows from the boundedness from above the ordinal types of well-ordered subsets of any fixed linear ordered set.

The theorem is proved. □

Definition 2.1. *The system of formal power series $\{\gamma_h \,|\, h \in G\}$ is called $\mathcal{T}$-basis if it is monotone, complete and σ-summable.*

Without lost of generality it can be assumed that for $\mathcal{T}$-basis the inequality $h_1 < h_2$ implies the inequality $v(\gamma_{h_1}) < v(\gamma_{h_2})$. This kind of monotonicity condition will be assumed in further. The following proposition give the examples of $\mathcal{T}$-bases.

Proposition 2.1. *For any nonzero element $r \in KG$ the system $\{r \cdot h \,|\, h \in G\}$ is $\mathcal{T}$-basis.*

Proof. Assume that $r = a_1 g_1 + \ldots + a_k g_k$, where $a_j \in K^*$ and $g_j \in G$ are pairwise distinct. For any $g \in G$ the following equality holds:

$$v(r \cdot g) = \min\{g_1 g, \ldots, g_k g\} \tag{2.5}$$

Prove the completeness of the family $\{rh\}$. Let $h \in G$ be arbitrary. Let choose the maximal element in the set $\{g_1^{-1}h, \ldots, g_k^{-1}h\}$. Without lost of generality it can be assumed that this element is $g = g_1^{-1}h$. Then g is the required element, that is $v(r \cdot g) = h$. Indeed, $g_i g \geq g_1 g = h$, because $g = g_1^{-1}h \geq g_i^{-1}h$ for any i. The completeness is prove.

Let prove the monotony. Let $h_1 < h_2$ be the elements from G and $v(r \cdot h_1) = g_i h_1$, $v(r \cdot h_2) = g_j h_2$. Then

$$g_i h_1 \leq g_j h_1 < g_j h_2,$$

because (2.5) and the order "$<$" withstands multiplication on the left. □

Note that we can write the formal power series (1.1) as $\gamma = \sum_{g\in G}^{\mathcal{T}} g k_g$. The same is true for any $\mathcal{T}$-basis $\{\gamma_h \,|\, h \in G\}$. Namely, by Theorem 2.1 we obtain that the same power series γ can be uniquely decomposed into the form $\gamma = \sum_{h\in G}^{\mathcal{T}} \gamma_h \tilde{k}_h$ ($\tilde{k}_h \in K$).

The concept of a $\mathcal{T}$-basis is associated with automorphisms of the linear space $K\{G\}_K$ with corresponding properties. The linear operator $q : K\{G\}_K \to K\{G\}_K$ is called *locally monotone*, if for any $g, h \in G$ the inequality $g < h$ implies the inequality of the norms $v(q[g]) < v(q[h])$.

An endomorphism q is called *locally surjective* if for any $h \in G$ there is an element $g \in G$ such that $v(q[g]) = h$.

If q is simultaneously locally monotone, locally surjective and continuous, then we say that q is an *$\mathcal{T}$-automorphism*.

The main result of the paper is the following theorem

Theorem 2.2.

1) *Every $\mathcal{T}$-automorphism is an automorphism of the space $K\{G\}_K$ and maps any $\mathcal{T}$-basis into $\mathcal{T}$-basis.*
2) *The inverse mapping to $\mathcal{T}$-automorphism is also $\mathcal{T}$-automorphism. Moreover, the set of all $\mathcal{T}$-automorphisms is a subgroup in a group of the automorphisms* $\mathrm{Aut}K\{G\}_K$.

3) *If* $\{\beta_h \mid h \in G\}$ *and* $\{\gamma_h \mid h \in G\}$ *are two* $\mathcal{T}$*-bases, then the mapping* $q : K\{G\} \to K\{G\}$*, given by*

$$\sum_{h\in G}^{\mathcal{T}} \beta_h k_h \to \sum_{h\in G}^{\mathcal{T}} \gamma_h k_h \tag{2.6}$$

is $\mathcal{T}$*-automorphism.*

First we prove two lemmas.

Lemma 2.3. *Let* $q : K\{G\} \to K\{G\}$ *be a map such that for any well-ordered subset* $M \subseteq G$ *the system of formal power series* $\{q[h] \mid h \in M\}$ *is* $\mathcal{T}$*-summable and the following equality holds*

$$q\left[\sum_{h\in M} hk_h\right] = \sum_{h\in M}^{\mathcal{T}} q[h]k_h$$

for any set of coefficients $k_h \in K$*. Then* q *is a continuous linear endomorphism.*

Proof. Let $\{\gamma_i \in K\{G\} \mid i \in I\}$ be $\mathcal{T}$-summable family. Then $N = \bigcup_{i\in I} \operatorname{supp} \gamma_i$ is a well-ordered set. From the condition it follows that $M = \bigcup_{h\in N} \operatorname{supp} q[h]$ is a well-ordered set too. Let

$$\gamma_i = \sum_{h\in N} hk_h^i \ \ (i \in I) \ \text{ and } \quad q[h] = \sum_{g\in M} gk_g^h \ \ (h \in N).$$

By definition $\mathcal{T}$-sum we have

$$q[\gamma_i] = \sum_{h\in N}^{\mathcal{T}} q[h]k_h^i = \sum_{g\in M} g\left(\sum_{h\in N} k_g^h k_h^i\right).$$

Then

$$\sum_{i\in I}^{\mathcal{T}} \gamma_i = \sum_{h\in N} h\left(\sum_{i\in I} k_h^i\right);$$

$$q\left[\sum_{i\in I}^{\mathcal{T}} \gamma_i\right] = \sum_{h\in N}^{\mathcal{T}} q[h]\left(\sum_{i\in I} k_h^i\right) = \sum_{g\in M} g\left(\sum_{h\in N} k_g^h \sum_{i\in I} k_h^i\right).$$

For fixed $g \in M$ almost all products $k_g^h k_h^i$ are equal to 0.

Indeed, $\mathcal{T}$-summability $\{q[h] \mid h \in N\}$ implies that there are only a finite number $h \in N$ such that $k_g^h \neq 0$. For each of these h there are only a finite number of indices i with $k_h^i \neq 0$, because $\{\gamma_i\}$ is a $\mathcal{T}$-summable system. Then

$$\sum_{h\in N, i\in I} k_g^h k_h^i = \sum_{h\in N} k_g^h \sum_{i\in I} k_h^i = \sum_{i\in I}\left(\sum_{h\in N} k_g^h k_h^i\right).$$

In addition, the union $\bigcup_{i\in I}\operatorname{supp} q[\gamma_i]$ is a subset of the well-ordered set M, hence it is well-ordered set too. Consequently, the system $\{q[\gamma_i]\}$ is $\mathcal{T}$-summable and

$$\sum_{i\in I}{}^{\mathcal{T}} q[\gamma_i] = \sum_{g\in M} g\left(\sum_{i\in I}\left(\sum_{h\in N} k_g^h k_h^i\right)\right) =$$
$$\sum_{g\in M} g\left(\sum_{h\in N} k_g^h \sum_{i\in I} k_h^i\right) = q\left[\sum_{i\in I}{}^{\mathcal{T}}\gamma_i\right].$$

By definition q it follows its homogeneity i.e an equality $q[\gamma k] = q[\gamma]k$ is true for all $\gamma \in K\{G\}$, $k \in K$.

Thus, the lemma is completely proved. □

Denote by $\hat{\Gamma}$ Dedekind closure of a linearly ordered set G. Any subset $\hat{\Gamma}\cup\{\infty\}$ has an least upper bound. If $\gamma \in K\{G\}$ and $\varepsilon \in \hat{\Gamma}\cup\{\infty\}$ then we denote by $(\gamma)_{<\varepsilon}$ ε the *beginning of the series* γ, i.e. formal power series β such that $\operatorname{supp}\beta < \varepsilon$ and $v(\gamma-\beta) \geq \varepsilon$. If

$$\gamma = (\gamma)_{<\varepsilon} + \delta, \tag{2.7}$$

then $\operatorname{supp}(\gamma)_{<\varepsilon} < \varepsilon \leq \operatorname{supp}\delta$. The decomposition (2.7) is called ε-*series section* of γ.

Lemma 2.4. *If q is monotone and continuous automorphism then q^{-1} is continuous.*

Proof. Suppose that q^{-1} is not a continuous endomorphism. Then by Lemma 2.3 there is a well-ordered subset $M \subseteq G$ such that the system $\{\gamma_h = q^{-1}[h] \mid h \in M\}$ is not $\mathcal{T}$-summable. Indeed, if the system $\{\gamma_h = q^{-1}[h] \mid h \in M\}$ is $\mathcal{T}$-summable, then the equality $q^{-1}[\sum_{h\in M}^{\mathcal{T}}\gamma_h] = \sum_{h\in M}^{L} q^{-1}[\gamma_h]$ is true because it is equivalent to equality

$$\sum_{h\in M}{}^{\mathcal{T}}\gamma_h = q\left[\sum_{h\in M}{}^{L} q^{-1}[\gamma_h]\right],$$

which is correctly by continuity q.

Denote by

$$\nu = \sup\left\{\varepsilon \in \hat{\Gamma}\cup\{\infty\} \mid \text{ system } \{(\gamma_h)_{<\varepsilon} \mid h \in M\} \text{ is } \mathcal{T}\text{-summable}\right\}.$$

Since $((\gamma)_{<\varepsilon})_{<\rho} = (\gamma)_{<\rho}$ for any $\varepsilon,\rho \in \hat{\Gamma}\cup\{\infty\}$, $\rho \leq \varepsilon$ then it can easily be checked that for any $\rho \in \hat{\Gamma}\cup\{\infty\}$ strictly less than ν, the system $\{(\gamma_h)_{<\rho} \mid h \in M\}$ is summable.

Moreover the system $\{(\gamma_h)_{<\nu} \mid h \in M\}$ is $\mathcal{T}$-summable. Indeed, let $h_1 < h_2 < h_3 < \ldots$ be $\mathbb{N}$-sequence of the elements from M, and the elements $z_i \in \operatorname{supp}(\gamma_{h_i})_{<\nu}$ $(i \in \mathbb{N})$ are chosen arbitrarily. Suppose that $z = z_1$ is strictly less than all other z_i, $i > 1$. Since $z \in \operatorname{supp}(\gamma_{h_1})_{<\nu}$ then $z < \nu$. Consequently, the system $\{(\gamma_h)_{<z} \mid h \in M\}$ is $\mathcal{T}$-summable. But $z_i \in \operatorname{supp}(\gamma_{h_i})_{<z}$ for $i > 1$, because $z_i < z$. Then $z_i < z_j$ for a pair of natural numbers $i < j$ (see proposition 1.1). If z is not a maximal element, then $z_1 < z_i$ for a suitable $i \in \mathbb{N}$. Thus the conditions of proposition 1.1 are fulfilled, this implies that the system $\{(\gamma_h)_{<\nu} \mid h \in M\}$ is $\mathcal{T}$-summable.

Denote by $\alpha_h = (\gamma_h)_{<\nu}$. Let $\gamma_h = \alpha_h + \beta_h$ be a ν-section. Since the system $\{\gamma_h\}$ is not $\mathcal{T}$-summable, and the system $\{\alpha_h\}$ is $\mathcal{T}$-summable as proved above then the system $\{\beta_h\}$ is not $\mathcal{T}$-summable. Moreover the set of norms $\{v(\beta_h) \mid h \in M\}$ is not well-ordered, because in the converse case there exist $g \in \bigcup_{h\in M} \operatorname{supp} \beta_h$ more than ν such that the systems $\{(\beta_h)_{<g} \mid h \in M\}$ and $\{(\gamma_h)_{<g} = \alpha_h + (\beta_h)_{<g} \mid h \in M\}$ are $\mathcal{T}$-summable, that contradicts the maximality of the element.

So the set $\{v(\beta_h) \mid h \in M\}$ is not well-ordered. Hence by monotonicity of the automorphism q it follows that $\{v(q[\beta_h]) \mid h \in M\}$ is not well-ordered too. On the other hand

$$q[\beta_h] = q[\gamma_h] - q[\alpha_h] = h - q[\alpha_h],$$

where $\{h \mid h \in M\}$ and $\{q[\alpha_h] \mid h \in M\}$ are $\mathcal{T}$-summable systems. Note the second system is $\mathcal{T}$-summable because automorphism q is σ-linear. So their difference is $\mathcal{T}$-summable. The resulting contradiction shows that our assumption was not correct. Consequently, the automorphism q^{-1} is continuous. □

Proof of the theorem 2.2.

1. Let q be an $\mathcal{T}$-automorphism. Then the system of formal power series $\{q[h] \mid h \in G\}$ is $\mathcal{T}$-basis because monotony and completeness of this system is exactly local monotonicity and local surjectivity of the endomorphism q, and σ-summability of the system follows from the continuity of q, if we assume $I = G$ in the definition of σ-summability Moreover

$$q\left[\sum_{h\in G}{}' hk_h\right] = \sum_{h\in G}^{\mathcal{T}} q[h]k_h \tag{2.8}$$

for any formal power series $\gamma = \sum_{h\in G}^{\mathcal{T}} hk_h$. By (2.8)it follows that q is monomorphism because $q[\gamma] = 0$ implies $k_h = 0$ for all $h \in G$. Applying theorem 2.1 to a system of formal power series $\gamma_h = q[h]$ $(h \in G)$ by (2.8) we get that q is epimorphism.

Further, let $\{\beta_h \mid h \in G\}$ be a $\mathcal{T}$-basis. Then the system $\{q[\beta_h] \mid h \in G\}$ is σ-summable, because the endomorphism q is continuous. The local monotony and the local surjectivity of q implies the monotony and completeness of this system because the equality

$$v(q[\gamma]) = v(q[v(\gamma)]) \tag{2.9}$$

is true for any formal power series γ and for any locally monotone and continuous endomorphism. Prove this statement. We can assume $\gamma \neq 0$. Then $\gamma = \partial\gamma + \delta$, where $\operatorname{supp}\delta > \operatorname{supp}(\partial\gamma) = \{v(\gamma)\}$. The local monotony of q implies $v(q[g]) > v(q[\partial\gamma])$ for any $g \in \operatorname{supp}\delta$. Then by the continuity of the endomorphism q we get $v(q[\delta]) > v(q[\partial\gamma])$. Since $q[\gamma] = q[\partial\gamma] + q[\delta]$, we have

$$v(q[\gamma]) = v(q[\partial\gamma]) = v(q[v(\gamma)]),$$

because $\partial\gamma = v(\gamma)k$ for some nonzero $k \in K$.

2. Let q be an $\mathcal{T}$-automorphism. From the inequality $v(q^{-1}[g]) \geq v(q^{-1}[h])$ for $g, h \in G$ and (2.9) it follows that $g \geq h$. So q^{-1} is locally monotone. By Lemma 2.4 it follows that the automorphism q^{-1} is continuous. Besides q^{-1} is locally surjective. Indeed, for any $h \in G$, the equality

$$h = v(q^{-1}[q[h]]) = v(q^{-1}[v(q[h])])$$

, because q^{-1} is continuous and locally monotone and (2.9). Therefore $g = v(q[h]) \in G$ is required element, i.e. $v(q^{-1}[g]) = h$.

It remains to show that the composition of two $\mathcal{T}$-automorphisms is $\mathcal{T}$-automorphism. From the first assertion of the theorem it follows that this composition maps the standard $\mathcal{T}$-basis G into $\mathcal{T}$-basis. A composition of continuous maps is a continuous map. Using equalities of the type (2.9) we get local monotony and local surjectivity. Consequently, the composition is a $\mathcal{T}$-automorphism.

3. Decompose the map (2.6) into a composition of two maps $q = s \circ p^{-1}$, where

$$p\left[\sum_{g\in G}{}' g k_g\right] = \sum_{g\in G}{}^{\mathcal{T}} \beta_g k_g; \quad s\left[\sum_{g\in G}{}' g k_g\right] = \sum_{g\in G}{}^{\mathcal{T}} \gamma_g k_g.$$

From lemma 2.3 all p and s are $\mathcal{T}$-automorphisms, so it remains to apply the second assertion of the theorem 2.2.

This completes the proof of theorem 2.2.

Corollary 2.1. *(see* [5]*). Let $r \in FG$ be a nonzero element. Then the multiplication on the left of the element r is $\mathcal{T}$-automorphism and the system $\{r^{-1}[g] \mid g \in G\}$ consisting of formal power series $r^{-1}[g]$ such that $r \cdot r^{-1}[g] = g$ is a $\mathcal{T}$-basis*

Moreover, for any formal power series γ of the form (1.1) there exist a formal power series

$$r^{-1}[\gamma] = \sum_{h\in G}{}^{L} r^{-1}[h] k_h$$

which is unique solution to the equation $r \cdot X = \gamma$.

Proof. According to the proposition 2.1 the system $\{r \cdot h \mid h \in G\}$ is $\mathcal{T}$-basis. It remains to apply Theorem 2.2. □

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